Algebraic geometry 1

Exercise Sheet 6

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Exercise 1. Let $f \in K[X_1, ..., X_n]$ be a polynomial of degree d. The homonization of f is defined as the following homogeneous polynomial

$$f^h := X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right) \in K[X_0, X_1, \ldots, X_n]\,.$$

For an ideal $J \in K[X_1, \ldots, X_n]$ we denote by J^h the ideal generated by f^h for all non-zero $f \in J$.
Let $X = V^a(J) \subset \mathbb{A}^n$ be an affine algebraic set. We can identify $\mathbb{A}^n$ with the open subset $U_0 := \{[x_0 : x_1 : \cdots : x_n] \mid x_0 \neq 0\} \subset \mathbb{P}^n$ by the bijective map

$$\mathbb{A}^n \to U_0, \quad (x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]\,.$$

Consider now X as a subset of $\mathbb{P}^n$ (that is $X \subset \mathbb{A}^n \subset \mathbb{P}^n$).

(1) Show that $\overline{X} = V^p(J^h)$ (with respect to the Zariski topology on $\mathbb{P}^n$).
(2) Let $f \in K[X_1, ..., X_n]$ and $X = V^a(f)$. Deduce from (1) that $\overline{X} = V^p(f^h)$.

Exercise 2. Let $Y := V^p(Y^2 - XZ) \subset \mathbb{P}^2$.
(1) Show that the map

$$\varphi : \mathbb{P}^1 \to Y, \quad [X_0, X_1] \mapsto [X_0^2, X_0X_1, X_1^2]\,,$$

is regular.
(2) Show that φ is an isomorphism (that is, φ is bijective and the inverse map is also regular).

Exercise 3. Let $Y_1 := V^a(X_2 - X_1^2)$ and $Y_2 := V^a(X_1X_2 - 1)$ be two affine algebraic varieties in $\mathbb{A}^2$.
Consider their closures $\overline{Y_1}$ and $\overline{Y_2}$ in $\mathbb{P}^2$ (as explained in Exercise 1).
Show that $\overline{Y_1} \simeq \overline{Y_2}$ but Y_1 and Y_2 are not isomorphic as affine varieties.

Exercise 4. Show that the only regular functions on $\mathbb{P}^n$ are the constant functions.